An Experienced Teacher’s Conceptual Trajectory for Problem Solving

Barbara Butterfield
University of Wollongong
<butterfi@uow.edu.au>

This study explores how teachers understand and translate curriculum statements into concepts that are better suited to supporting student progress with rich mathematical problem-solving tasks. We report the actions of one experienced teacher of primary mathematics in charting the sequence of concepts and processes relevant to the above issue. We contend that the study embellishes Pedagogical Content Knowledge dimensions of the framework advanced by Ball, Hill and Bass (2005). Implications for field experiences of prospective teachers of mathematics are discussed.

Background

The need to be innovative about programs that are used to support prospective teachers of mathematics continues to be a challenge for tertiary educators. The task is complicated by the ever changing demands of school environments, precipitated by a mobile student and teacher population that is expected to function in a competitive globalised world. In such fluid classroom environments, teachers have to contend with meeting professional standards of performance (NSW Institute of Teachers, 2005), as well as curriculum reforms such as the adoption by the states of the National Curriculum (ACARA, 2010) Mathematics curriculum documents, among other things, identify core content areas and learning outcomes for teachers. It is a teacher’s responsibility to translate the statements of content and learning outcomes into appropriate teaching actions and learning activities in the school environment. Against this background, what is the nature of a teacher’s knowledge base that is required in making the transition from curriculum statements to action in the classroom?

We explored the above question initially by examining the quality of knowledge that pre-service teachers (PSTs) could bring to the design, solution and teaching of mathematics problems for young children (Authors, 2010, 2011). The results of these studies indicated that PST’s tend to have built a reasonably robust body of mathematical content knowledge. However, as expected, they experienced difficulties in comprehending and translating curriculum statements into designing of learning activities that are rich and engaging and showed a degree of alignment with the spirit of the curriculum statements. In particular, we found that PSTs could not unpack the learning outcomes as presented in the K-6 mathematics curriculum (Board of Studies, 2006) in ways that would foreground the underlying mathematical concepts in the design of learning tasks.

In order to better understand the above issue, we sought to draw on studies of teacher expertise with the view to generating information about what this group of teachers faced regarding content and learning outcome statements that appear in K-6 mathematics curriculum. Expert-novice studies of mathematics teachers and teaching have made important inroads into our understanding of teacher cognition and the psychology of mathematics teachers (Chinnappan & Lawson, 2005; Leinhardt, 1987, 1988, 1989; Schoenfeld, 2000). A significant finding from this stream of research is that teachers draw

In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia). Singapore: MERGA.

on robust content knowledge but this knowledge has to be utilised flexibly in supporting learning. This point has been foregrounded in recent works by Ball and colleagues at the University of Michigan.

Objective

The aim of this study was to understand an aspect of mathematics teachers' Pedagogical Content Knowledge (PCK) in the domain of primary mathematics, by describing the knowledge and actions of one experienced primary mathematics teacher as she attempted to develop a problem-solving task on the basis of a K-6 Mathematics curriculum that is adopted in the state of New South Wales (NSW), Australia.

Theoretical Framework

Data analysis and interpretations of our participant were guided by a model of Mathematical Knowledge for Teaching proposed by Ball, Bass, and colleagues (e.g., Ball et al., 2008). Under this model, Mathematical Knowledge for Teaching (MKT) is comprised of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). Pedagogical Content Knowledge is knowledge of suitable teaching strategies for the content, including knowledge of how to arrange components of the content for effective teaching. Subject Matter Knowledge (SMK), on the other hand, is knowledge of content that is not necessarily related to the way that mathematics could be taught. Both types of knowledge are considered relevant to the study of mathematics in the education context, and the division of SMK and PCK allows research to focus in each domain. A diagrammatic representation of the Mathematical Knowledge for Teaching (MKT) model is depicted in Figure 1.

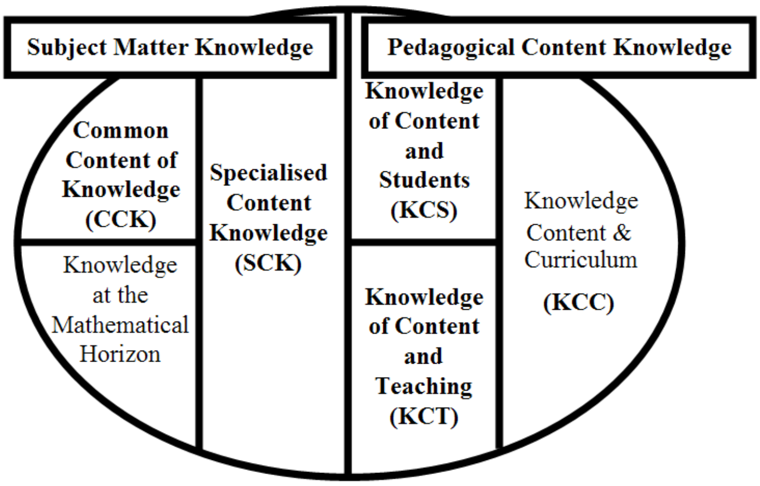

Figure 1. Schematic representation of teacher knowledge for teaching mathematics (Hill, Ball & Schilling, 2008: 174).

In the present study, we focus on the domain of Pedagogical Content Knowledge (PCK). Three dimensions are identified within PCK (see Figure 1).

Knowledge of Content and Students (KCS)
Knowledge that combines knowing about mathematics and knowing about students. Knowledge of how to: anticipate what students are likely to think; relate mathematical ideas to developmentally appropriate language used by children.

Knowledge of Content and Teaching (KCT)
Knowledge that combines knowing about mathematics and knowing about teaching. Knowledge of how to: sequence content for instruction; determine instructional advantages of different representations; pause for clarification and when to ask questions; analyse errors; observe and listen to a child's responses; prompt, pose questions and probe with questions; select appropriate tasks.

Knowledge of Content and Curriculum (KCC)
In a more recent model (Ball, 2010), the dimension of Knowledge of Curriculum was developed into *Knowledge of Content and Curriculum* (*KCC*) thus emphasising the link between mathematical content knowledge and understanding of mathematics curriculum including statements of learning outcomes, assessment strategies and lesson planning sequences as well as the development of learning sequences that are appropriate for different grade levels.

In this paper we provide data to indicate links between knowledge of mathematics curriculum KCC, KCS and KCT as demonstrated by an experienced teacher.

Method

Design and Participant:

The research design was a case study involving an experienced teacher of primary mathematics. The teacher was a volunteer with 25 years of classroom experience and had been judged by her peers to be an outstanding practitioner in the state of NSW, Australia. This teacher will be referred to as Julia.

Procedure

Julia was invited to participate in two, two-hour interviews. Semi-structured interview questions were used to generate data in this present study. During the first interview, Julia was informed about the purpose and objectives of the study. At this interview Julia was asked to think about how she uses the NSW K-6 Mathematics (2002) curriculum for lesson planning and teaching. Specifically, Julia was encouraged to design a rich learning task that would be appropriate for children in Year 6 (11 year olds) in Australia. During the second interview, Julia was asked to talk about the major strands of the curriculum and links to the learning task that she had designed.

Data Sources

The primary data source for the present study was responses from Julia to the semi-structured interview questions. Responses that are relevant to this study of Pedagogical Content Knowledge are reported in the Results and Conclusions section - Rich Learning

Task (Truss Bridge problem Figure 2), Conceptual Trajectory (Table 1) and Learner Scaffoldings (Table 2).

Results and Conclusions

Rich Learning Task

The Rich Learning Task that was designed consisted of the photograph shown in Figure 2, followed by a set of questions relating to numbers of paddle pop sticks required for the side of this bridge, and the pattern that describes how many paddle pop sticks would be required for any length bridge. The questions also encourage thinking about other situations that may use this type of pattern.

Figure 2. Truss Bridge Problem (TBP)

The solution of the problem involves integration of content knowledge from a number of strands such as number, geometry and algebra, and use of a range of problem-solving strategies, such as reasoning and pattern identification. We see the richness of the Truss Bridge Problem as providing evidence of Julia accessing and integrating content knowledge, in a manner that will encourage students to think deeply about the application of mathematics in a real-life context. This demonstrates the teacher's activation of KCT and KCS.

During the second interview, Julia responded to our questions about the major strands of the curriculum and links to the learning task that she had designed. In order to examine linkage between Knowledge of Content and Curriculum (KCC) and Knowledge of Content and Teaching (KCT) we cross-correlated Julia's responses regarding Content and Process to the curriculum Content and Process Statements. The set of Content and Process data obtained is included as a Conceptual Trajectory in Table 1.

In Table 1, curriculum statements are labelled as 1aES1, 2aES1, 2aS3 and so on. For example, 1aES1 refers to a curriculum statement involving a geometry concept. The corresponding label for algebra is 2aES1. The translations by Julia are labelled as 1bES1, 2bES1 and so on. We note a number of connections in the participant's translations of the curriculum statements to the planned task. Firstly, Julia is very focussed on and explicit about the geometric and algebraic concepts that are played out at each of the Learning Stages (BOS, 2002) for children. For example, for children in Early Stage One (ES1), the curriculum states that 'Students learn about comparing and describing closed shapes and open lines' in the area of geometry (1aES1). This is interpreted by Julia as 'Identifies triangles in pictures and in real-life contexts' (1bES1) in the context of the Truss Bridge Problem. Secondly, she demonstrates the sequencing of concepts as she moves from Early

Stage One (ES1) to Stage Three (S3). Thus, within each Stage of learning Julia attempts to translate the corresponding curriculum statement into a form that is required and used in understanding and solving the Truss Bridge Problem. We contend that the identified activation of Knowledge of Content and Curriculum (KCC) and Knowledge of Content and Teaching (KCT) in this context demonstrates this mathematics teacher's Pedagogical Content Knowledge (PCK) in the domain of primary mathematics.

Table 1
Conceptual Trajectory

	Content and processes statements as per the K-6 syllabus (Board of Studies, 2006)		**Julia's translations of the K-6 syllabus content and processes for the designed task**	
	Geometry	Algebra	Geometry	Algebra
Early Stage (ES1)	1aES1 - Students learn about comparing and describing closed shapes and open lines	2aES1 – Students learn to ask questions about how repeating patterns are made and how they can be copied on continued	1bES1 – Students identify triangles in pictures and in real-life contexts.	2bES1 – Students recognise, copy and continue repeating patterns using shapes e.g.∇Δ∇Δ
Stage 1 (S1)	1aS1 – Students learn about sorting 2-Dimensional shapes by a given attribute.	2aS1 – Students learn to represent number patterns using diagrams, words, symbols	1bS1- Student identify sides and corners of triangles	2bS1 – Students determine a missing element in a number pattern e.g. 3, 6, 9, ☐, 15
Stage 2 (S2)	1aS2 – Students learn to recognise that a particular shape can be represented in different sizes and orientations	2aS2 – Students learn to pose problems based on number patterns	1bS2 – Student make representations of triangles in different orientations	2bS2- Students make a number pattern using trusses and pose a problem based on that pattern
Stage 3 (S3)	1aS3 – Students learn to explain classification of 2-D shapes	2aS3 – Students learn to use a number of strategies to solve unfamiliar problems	1bS3 – Students compare and describes side properties of equilateral, isosceles and scalene triangles	2bS3 – Students work through a process of building a truss with a series of equilateral triangles, to determine a rule that links the number of trusses and length of any bridge

We also investigated the activation of Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) during the second interview conducted with Julia. Our interview generated response data, consisting of information about potential areas of difficulty that could be encountered by the children while solving the Truss Bridge Problem questions. These data are presented as Learner Scaffolding in Table 2.

Table 2
Learner Scaffolding

<table>
<tr><th>Likely Difficulties for Students</th><th>Useful Representations to Aid Understanding</th></tr>
<tr><td>Interpreting a 3D object in a photograph to visualise a 2D side view (elevation).</td><td>Photo with triangles highlighted</td></tr>
<tr><td>Knowledge of equilateral triangles (beams are equal length).</td><td>Construct equilateral triangles with paddlepop sticks</td></tr>
<tr><td>Recognition of an inverted triangle</td><td>Manipulation of paper triangles to form alternate pattern</td></tr>
<tr><td>Recognition of series of beams</td><td>Highlight the beams in different colours
Build the beams in different colours</td></tr>
<tr><td>Lack of knowledge of a truss</td><td>View real world examples such as crane, bridge, power line towers</td></tr>
<tr><td>Cannot conceptualise shared side in truss</td><td>Highlight shared side between two triangles</td></tr>
<tr><td>Cannot visualise how the pattern could continue</td><td>Draw on the photograph
Build the side using paddlepop sticks</td></tr>
<tr><td>Does not know how to count the beams</td><td>Touch each stick and count.
Draw the side view. Mark each side while counting (one to one correspondence).</td></tr>
<tr><td>Does not know how to record patterns clearly and/or efficiently.</td><td>Draw table and demonstrate use of the table (students can use supporting representations e.g. drawings, paddlepop stick model to complete the table).<table><tr><td>N^{o} Δ</td><td>1</td><td>2</td><td>3</td></tr><tr><td>N^{o} sides</td><td>3</td><td>5</td><td>7</td></tr></table></td></tr>
<tr><td>Cannot determine a rule for the pattern from the table</td><td>Start with one beam and show that every new triangle requires two more beams
one + twice the number of triangles</td></tr>
</table>

The left-hand column of Table 2 shows Julie's anticipated possible student difficulties, while in the right-hand column are Julie's suggestions to assist and support students. For example, if children experience difficulty with recognition of an inverted triangle, Julia suggested that teachers could provide an activity where children could experiment with the different orientations of a triangle (Manipulation of paper triangles to form alternate pattern). We contend that these responses demonstrate activation of KCS and KCT that was triggered by Julia's analysis of difficulties children may experience when solving the Truss Bridge Problem (TBP). There is evidence of our participant decomposing and integrating elements of the TBP. Julie clearly provided an in-depth analysis of potential challenges in solving the Truss Bridge Problem, and appropriate assistance in the form of representations to aid understanding. In addition, Julie alluded to the necessity of timely support for individual children at their point of need. A result in the classroom of this activation of KCT could be effective time management, as the children could be given suitable time to individually experiment with important concepts and their representations, both in concrete and abstract forms.

Discussion

In advancing frameworks of Mathematical Knowledge for Teaching, Ball (2010) has shown the importance of teachers connecting their Subject Matter Knowledge and Pedagogical Content Knowledge. This study is an attempt to provide insight into links between the components of Pedagogical Content Knowledge (PCK). In designing a rich learning task an experienced teacher draws on Knowledge of Content and Curriculum (KCC) and Knowledge of Content and Teaching (KCT). These are components of Pedagogical Content Knowledge. This study also identifies that there are links between Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT), demonstrated through the development of appropriate learner scaffolding in the context of designing a rich learning task.

The translation of mathematical content knowledge by the teacher is evidenced by the development of a rich learning task and in articulating the pedagogical significance of the task in light of its conceptual underpinnings. This translation lies at the heart of teachers having to restructure their knowledge, which featured strongly in arguments developed by Putnam (1987) and Tamir (1988) about knowing mathematical content in different forms that are germane to teaching.

Although our study was limited to one teacher's knowledge and actions, there are lessons here for tertiary programs that aim to prepare future teachers of primary mathematics (Mewborn, 2001). Specifically, we contend that in supporting pre-service teachers we need to go beyond assessing their content knowledge of mathematics and provide learning and field experiences where they are immersed in activities that enable them to a) discuss curriculum statements, b) draw out the relevant mathematical concepts and c) identify areas of difficulty. Such experiences could then be utilised in the design of rich learning tasks for children. A closer examination into relations between the components of Mathematical Knowledge for Teaching is required in order to support Pre-Service mathematics teachers with innovative programmes.

References

ACARA (2010). *Report on Trial School Consultation* . Australian Curriculum, Assessment and Reporting Authority. http://www.acara.edu.au/verve/_resources/Report_on_Trial_School_Consultation.pdf

Ball, D. (2010). *Knowing mathematics well enough to teach it: From teachers' knowledge to knowledge for teaching*. http://www-personal.umich.edu/~dball/presentations/042710_ISRC.pdf.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 5*(3), 43-46.

Butterfield, B. & Chinnappan, M. (2011). Teacher Knowledge Activated in the Context of Designing Problems. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics Traditions and [New] Practices AAMT/MERGA Conference 2011* (pp. 142-150). Alice Springs N.T.: AAMT & MERGA.

Butterfield, B. & Chinnappan, M (2010). Walking the talk: Translation of mathematical content knowledge to practice. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education.Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 109–116). Fremantle, WA: MERGA.

Board of Studies (2006). *Mathematics K-6 Syllabus*. Sydney: Board of Studies New South Wales.

Chinnappan, M. & Lawson, M. (2005). A framework for analysis of teachers' geometric content knowledge and geometric knowledge for teaching. *Journal of Mathematics Teacher Education*, *8*(3), 197–221.

Department of Education, Employment and Workplace Relations (2008). *Review of Australian Higher Education*. Author, http://www.deewr.gov.au/highereducation/review/pages/reviewofaustralianhighereducationreport.aspx

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching, 15(1),* 13-33.

Hill, H. C., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372–400.

Leinhardt, G. (1988). Expertise in instructional lessons: An example from fractions. In D. A. Grouws & R .J. Cooney (Eds.) *Perspectives on research on effective mathematics teaching* (Vol. 1 in a series of monographs from conferences of the National Council for Teachers of Mathematics, pp. 47-76). Hilldale, NJ: Erlbaum.

Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal for Research in Mathematics Education*, *20*(1), 2-75.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers. *United States Mathematics Education Research Journal, 3*, 28–36.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA.: Author

NSW Institure of Teachers (2005), *Professional Teaching Standards.* http://www.nswteachers.nsw.edu.au/Main-Professional-Teaching-Standards/

Putnam, R.T. (1987). Structuring and adjusting content for students: A study of live and simulated tutoring of addition. *American Educational Research Journal*, *24* (1), 13-28.

Schoenfeld, A. H. (2000). Models of the Teaching Process. *Journal of Mathematical Behavior, 18*(3), 243-261.

Tamir, P. (1988). Subject matter and related pedagogical knowledge in teacher education. *Teaching and Teacher Education*, *2,* 99-110.